Journal of Engineering Mathematics **42:** 119–131, 2002.

TE-impedance of a coated right-angled wedge

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Received 7 August 2001; accepted in revised form 23 November 2001

Abstract. The surface impedance observed by a plane TE-wave impinging on a coated right-angled perfectly conducting wedge is derived for a high contrast dissipative coating. The impedance proves to be constant ove rmost of the surface of the coating. The value of the constant agrees with that obtained when the coating is placed on an infinite perfectly conducting plane. Near the edge of the coating, however, the impedance is not invariable. Both the magnitude and phase can deviate substantially from their asymptotic values; also they change with the angle of incidence of the irradiating wave. The region of variability depends on the amount of absorption but does not exceed a free-space wavelength for the cases considered.

Key words: dielectric, diffraction, electromagnetism, scattering theory, wedge

1. Introduction

Coated surfaces are not uncommon obstacles encountered by electromagnetic waves. As the frequency grows, the influence of the coating becomes more important. With mobile phones, to mention one example, propagation in urban areas can involve interaction with a variety of structures. To aid our understanding of what happens, there is a need to tackle canonical problems. Canonical problems not only offer some insight into phenomena but also serve as benchmarks for the approximate methods which have to be deployed in practical situations.

One canonical problem which has defied analytical solution so far is the coated wedge. Indeed, it is only recently that some progress has been made with the dielectric wedge (see Jones [1, 2] and the references cited therein). It would be helpful if the coated wedge could be modelled by means of an impedance boundary condition on its faces. Consideration of the interior of the coating could be avoided then. Also, the possibility arises of being able to apply the analytical and numerical methods available for wedges with impedance boundary conditions.

In this paper is considered the problem of a coated wedge in which the angle of the wedge is 90°. It is required that the coating be of high contrast and possess absorption. It is shown that the surface impedance is constant, except within a short distance (less than a free-space wavelength) of the edge. Thus a first approximation to the scattered field can be derived by treating the surface impedance as constant everywhere. Moreover, since there is no reason to suppose that the behaviour is peculiar to the right-angled wedge, it offers the opportunity of obtaining a scattering approximation for a wedge of any (non re-entrant) angle. Of course, neglect of the variation in the surface impedance near the edge may have an effect on the diffracted waves, but whether or not this is of significance is an open question.

When the point of observation is on the illuminated face well away from the edge, the coating behaves as if it were on an infinite plane. Therefore, the impedance there can be determined by standard techniques. It is this impedance which proves to be the constant referred to above. What is really remarkable is that the same constant occurs for the face in shadow. This could not have been expected. A face in shadow is not reached directly by the incident field. Instead its excitation comes from interior waves and fields diffracted at the edges. So it is not immediately evident that the face behaves as if it were backed by an infinite plane in the incident wave.

What the following investigation shows is that the surface impedance does coincide with that of an infinite plane interface, whether the face is in shadow or not, provided the point of observation is not too close to the edge. How far one has to be from the edge for the impedance to be constant depends on the dissipation: the larger it is, the shorter the distance has to be. The distance does not exceed a free-space wavelength for quite moderate absorption. While the impedance does not vary away from the edge, the situation near the edge is totally different. Here the behaviour of the impedance depends on the position of the exciting source in a complicated fashion. There can be profound differences in both magnitude and phase. Whether or not the face is in shadow affects the variation too.

The determination of a good approximation to the field in the coating when the structure is irradiated by a line source is formulated as a boundary-value problem in Section 2. The formulation assumes that the coating is absorbing and of high contrast. In Section 3 the boundary-value problem is converted into solving an integral equation. A method for the solution of the integral equation is developed in Section 4. The resulting field on the boundary of the coating is derived in Section 5 and specialised to the case of an incident plane wave in Section 6. An application of the theory to a particular example is given in Section 7.

2. Formulation

The cross-section of the coated wedge lies in the (x, y)-plane with the apex at the origin. The faces of the coating, which is right-angled, occupy $x \geq 0$ and $y \geq 0$, respectively. The thickness of the coating is taken to be h so that the faces of the perfectly conducting wedge are $x \geq h$, $y = h$ and $x = h$, $y \geq h$, respectively. The remainder of the (x, y)-plane is assumed to be free space.

Only fields which do not vary in the direction parallel to the edge of the coated wedge will be discussed. The dependence on time will be taken as $e^{i\omega t}$ and this factor will be suppressed subsequently. The wavenumber in free space is denoted by k. Inside the coating the wavenumber is k_1. The coating is assumed to be lossy so that k_1 has a negative imaginary part; as a consequence $k_1 = k_r - ik_i$ with both k_r and k_i positive. It will be assumed that the properties of the coating are such that $|k_1/k|^2 \gg 1$ holds.

Let the excitation be due to a line source, parallel to the edge of the wedge, located at the point (x_0, y_0) in free space. Then the field $p(x, y)$ in free space satisfies

$$(\nabla^2 + k^2)p(x, y) = -\delta(x - x_0)\delta(y - y_0), \tag{1}$$

where ∇^2 is the two-dimensional Laplacian. The field $p_1(x, y)$ inside the coating is a solution of

$$(\nabla^2 + k_1^2)p_1(x, y) = 0. \tag{2}$$

The boundary conditions on the faces of the coating are assumed to be

$$p = p_1, \qquad \frac{\partial p}{\partial n} = \frac{k^2}{k_1^2}\frac{\partial p_1}{\partial n} \tag{3}$$

$\partial/\partial n$ being a derivative normal to the boundary. On the wedge the boundary condition is

$$\partial p_1/\partial n = 0. \tag{4}$$

Furthermore, the field is required to be radiating at infinity. These conditions are pertinent, in electromagnetic applications, to H-polarised or TE-waves in which the permeability of the coating on the perfectly conducting wedge is the same as that of free space.

Let $G_0(x, y, \xi, \eta)$ be the Green's function in free space such that its normal derivative vanishes on the faces of the coating and which obeys the condition of radiating at infinity. It is a solution of

$$(\nabla^2 + k^2)G_0(x, y, \xi, \eta) = -\delta(x - \xi)\delta(y - \eta).$$

Denote by C the trace of the faces of the coating on the (x, y)-plane. On C let s be the arc length running from $-\infty$ to 0 on $x = 0$ and from 0 to ∞ on $y = 0$. Then, for (x, y) in free space,

$$p(x, y) = G_0(x, y, x_0, y_0) + \int_C G_0(x, y, \xi, \eta)\frac{\partial}{\partial n_\xi} p(\xi, \eta)\, \mathrm{d}s. \tag{5}$$

The direction of the unit normal n_ξ is into the coating so that $\partial/\partial n_\xi$ has the same sense as $\partial/\partial\xi$ on $x = 0$ and as $\partial/\partial\eta$ on $y = 0$.

As $|k_1/k| \to \infty$ the conditions in (3) indicate that $\partial p/\partial n \to 0$. It follows from (5) that $p(x, y) \to G_0(x, y, x_0, y_0)$. Therefore, a good first approximation when $|k_1/k|^2 \gg 1$ is

$$p(\xi, \eta) = G_0(\xi, \eta, x_0, y_0). \tag{6}$$

As a consequence of (3)

$$p_1(\xi, \eta) = G_0(\xi, \eta, x_0, y_0) \tag{7}$$

for (ξ, η) on the faces of the coating. Thus, a first approximation to p_1 is provided by a solution of (2) in the coating which complies with the boundary conditions of (4) and (7).

Once p_1 has been found $\partial p_1/\partial n$ is known on the faces of the coating and a correction to p is available from (5) via (3). Clearly, an iteration procedure could be formed on this basis. However, this avenue will not be explored because the main difficulty is the determination of p_1. Instead attention will be confined to finding p_1 subject to the boundary conditions (4) and (7).

3. Determination of p_1

Let $G_1(x, y, \xi, \eta)$ be the Green's function which satisfies

$$(\nabla^2 + k_1^2)G_1(x, y, \xi, \eta) = -\delta(x - \xi)\delta(y - \eta)$$

and complies with the boundary conditions

$$G_1(x, y, \xi, \eta) = 0$$

for $x \geq 0$, $y = 0$ and $x = 0$, $0 \leq y \leq h$ as well as

$$\frac{\partial}{\partial y}G_1(x, y, \xi, \eta) = 0$$

on $x \geq 0$, $y = h$. In addition, G_1 is to be exponentially decaying as $x \to \infty$. Then, in view of (3) and (4),

$$p_1(x, y) = \int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} G_1(x, y, \xi, h)\, \mathrm{d}\xi + \int_0^h p(0, \eta)\left[\frac{\partial}{\partial \xi} G_1(x, y, \xi, \eta)\right]_{\xi=0} \mathrm{d}\eta$$
$$+ \int_0^\infty p(\xi, 0)\left[\frac{\partial}{\partial \eta} G_1(x, y, \xi, \eta)\right]_{\eta=0} \mathrm{d}\xi \tag{8}$$

for $x \geq 0$, $0 \leq y \leq h$.

Next, let $G_2(x, y, \xi, \eta)$ be the Green's function satisfying

$$(\nabla^2 + k_1^2) G_2(x, y, \xi, \eta) = -\delta(x - \xi)\delta(y - \eta)$$

which meets the boundary conditions

$$G_2(x, y, \xi, \eta) = 0$$

on $x = 0$, $y \geq h$,

$$\frac{\partial}{\partial y} G_2(x, y, \xi, \eta) = 0$$

on $0 \leq x \leq h$, $y = h$ and

$$\frac{\partial}{\partial x} G_2(x, y, \xi, \eta) = 0$$

for $x = h$, $y \geq h$. Moreover, G_2 is to be exponentially decaying as $y \to \infty$. Then

$$p_1(x, y) = \int_h^\infty p(0, \eta)\left[\frac{\partial}{\partial \xi} G_2(x, y, \xi, \eta)\right]_{\xi=0} \mathrm{d}\eta - \int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} G_2(x, y, \xi, h)\, \mathrm{d}\xi \tag{9}$$

for $0 \leq x \leq h$, $y \geq h$.

Both p_1 and $\partial p_1/\partial \eta$ are continuous across the interface $0 \leq x \leq h, y = h$. These conditions are satisfied by (8) and (9) provided that

$$\int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} \{G_1(x, h, \xi, h) + G_2(x, h, \xi, h)\}\, \mathrm{d}\xi = f(x), \tag{10}$$

where

$$f(x) = \int_h^\infty p(0, \eta)\left[\frac{\partial}{\partial \xi} G_2(x, h, \xi, \eta)\right]_{\xi=0} \mathrm{d}\eta - \int_0^h p(0, \eta)\left[\frac{\partial}{\partial \xi} G_1(x, h, \xi, \eta)\right]_{\xi=0} \mathrm{d}\eta$$
$$- \int_0^\infty p(\xi, 0)\left[\frac{\partial}{\partial \eta} G_1(x, h, \xi, \eta)\right]_{\eta=0} \mathrm{d}\xi. \tag{11}$$

Since $f(x)$ is known, on account of (6), Equation (10) constitutes an integral equation on the interval $(0, h)$ of x for $\partial p_1/\partial \eta$. Once this integral equation has been solved, p_1 is known throughout the coating by means of (8) and (9).

4. Solution of the integral equation

In order to solve the integral equation (10) explicit expressions are needed for G_1 and G_2. Let $\nu_n = (n + \frac{1}{2})\pi/h$ and $\kappa_n^2 = k_1^2 - \nu_n^2$. Define κ_n to have a negative imaginary part; this is consistent with the definition of k_1. Then all the conditions imposed on G_1 and G_2 in the preceding section are met by

$$G_1(x, y, \xi, \eta) = \sum_{n=0}^{\infty} \frac{1}{\mathrm{i}\kappa_n h}\{\mathrm{e}^{-\mathrm{i}\kappa_n|x-\xi|} - e^{-\mathrm{i}\kappa_n(x+\xi)}\} \sin\nu_n y \sin\nu_n\eta, \tag{12}$$

$$G_2(x, y, \xi, \eta) = \sum_{n=0}^{\infty} \frac{1}{\mathrm{i}\kappa_n h}\{\mathrm{e}^{\times\mathrm{i}\kappa_n|y-\eta|} + e^{-\mathrm{i}\kappa_n(2h-y-\eta)}\} \sin\nu_n \sin\nu_n\xi. \tag{13}$$

The negative imaginary part of κ_n ensures that there is appropriate decay as $x \to \infty$ and $y \to \infty$, respectively, in the two cases.

Now multiply both sides of (10) by $\sin\nu_m x$ and integrate with respect to x from 0 to h. There results

$$\int_0^h \left[\frac{\partial p_1}{\partial\eta}(\xi, \eta)\right]_{\eta=h} \left\{\frac{1}{\kappa_m}(\tan\kappa_m h - \mathrm{i})\sin\nu_m\xi - \frac{2\mathrm{i}}{h}(-1)^m \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\kappa_n h}\sin\kappa_n\xi}{\nu_m^2 - \kappa_n^2}\right\} \mathrm{d}\xi$$

$$= \int_0^h f(x)\sin\nu_m x \,\mathrm{d}x \tag{14}$$

after taking advantage of the fact that

$$\sum_{n=0}^{\infty} \frac{1}{\nu_m^2 - \kappa_n^2} = \frac{h}{2\kappa_m}\tan\kappa_m h. \tag{15}$$

Let $p_1' = [\partial p_1(0, \eta)/\partial\eta]_{\eta=h}$. Then

$$\int_h^x \left\{\left[\frac{\partial p_1}{\partial\eta}(\xi, \eta)\right]_{\eta=h} - p_1'\right\} \mathrm{d}\xi$$

vanishes at $x = h$ and its derivative is zero at $x = 0$. Hence it can be expanded in terms of the orthogonal functions $\cos\nu_r x$. Suppose that, on $0 \le x \le h$,

$$\int_h^x \left\{\left[\frac{\partial p_1}{\partial\eta}(\xi, \eta)\right]_{\eta=h} - p_1'\right\} \mathrm{d}\xi = \sum_{r=0}^{\infty} d_r \cos\nu_r x \tag{16}$$

so that

$$\frac{1}{2}hd_r = \int_0^h \cos\nu_r x \int_h^x \left\{\left[\frac{\partial p_1}{\partial\eta}(\xi, \eta)\right]_{\eta=h} - p_1'\right\} \mathrm{d}\xi \,\mathrm{d}x.$$

An integration by parts gives

$$\nu_r d_r = \frac{2}{h}\left(\frac{p_1'}{\nu_r} - c_r\right), \tag{17}$$

where

$$c_r = \int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} \sin \nu_r \xi \, \mathrm{d}\xi. \tag{18}$$

On the other hand

$$\int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} \sin \kappa_n \xi \, \mathrm{d}\xi = \int_0^h \left\{\left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} - p_1'\right\} \sin \kappa_n \xi \, \mathrm{d}\xi$$
$$+\frac{p_1'}{\kappa_n}(1 - \cos \kappa_n h).$$

Integrate by parts and then substitute the representation in (16). It follows that

$$\int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} \sin \kappa_n \xi \, \mathrm{d}\xi = \frac{p_1'}{\kappa_n}(1 - \cos \kappa_n h) - \kappa_n \cos \kappa_n h \sum_{r=0}^{\infty} \frac{(-1)^r \nu_r d_r}{\nu_r^2 - \kappa_n^2}.$$

Replace d_r by means of (17) and observe that

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{\nu_r(\nu_r^2 - \kappa_n^2)} = \frac{h}{2\kappa_n^2}(\sec \kappa_n h - 1). \tag{19}$$

Consequently

$$\int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} \sin \kappa_n \xi \, \mathrm{d}\xi = \frac{2\kappa_n}{h} \cos \kappa_n h \sum_{r=0}^{\infty} \frac{(-1)^r c_r}{\nu_r^2 - \kappa_n^2}. \tag{20}$$

Insertion of (18) and (20) into (14) leads to

$$\frac{c_m}{\kappa_m}(\tan \kappa_m h - \mathrm{i}) - \frac{4\mathrm{i}}{h^2} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+r} \kappa_n c_r \, \mathrm{e}^{-\mathrm{i}\kappa_n h} \cos \kappa_n h}{(\nu_r^2 - \kappa_n^2)(\nu_m^2 - \kappa_n^2)} = \int_0^h f(x) \sin \nu_m x \, \mathrm{d}x, \tag{21}$$

which provides a linear system of equations to fix the coefficients c_r. Solution of the system determines $\partial p_1/\partial n$ in essence either through (18) or (16) and (17).

In fact, knowledge of the c_r is sufficient to determine the behaviour of p_1 throughout the coating. This point can be illuminated by finding the contribution of $\partial p_1/\partial \eta$ to $\partial p_1/\partial y$ on $y = 0$. This requires

$$I_1 = \left[\frac{\partial}{\partial y} \int_0^h \left[\frac{\partial p_1}{\partial \eta}(\xi, \eta)\right]_{\eta=h} G_1(x, y, \xi, h) \, \mathrm{d}\xi\right]_{y=0}.$$

When $x \geq h$ benefit can be drawn from (20) and

$$I_1 = \frac{4}{h^2} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{\eta+r} \frac{c_r \nu_n}{\nu_r^2 - \kappa_n^2} \cos \kappa_n h \, \mathrm{e}^{-\mathrm{i}\kappa_n x}. \tag{22}$$

For $0 \leq x \leq h$ the analysis is somewhat more complicated because of the structure of G_1. However, the pattern adopted in deriving (20) can be followed. Then, by virtue of (19) and

$$\sum_{r=0}^{\infty} \frac{\sin \nu_r x}{\nu_r(\nu_r^2 - \kappa_n^2)} = \frac{h}{2\kappa_n^2}(\cos \kappa_n x + \sin \kappa_n x \tan \kappa_n h - 1), \tag{23}$$

$$I_1 = \frac{2}{h}\sum_{r=0}^{\infty} c_r \frac{\sin \nu_r x}{\cos \kappa_r h} - \frac{4\mathrm{i}}{h^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{n+r}\frac{c_r \nu_n}{\nu_r^2 - \kappa_n^2}\mathrm{e}^{-\mathrm{i}\kappa_n h}\sin \kappa_n x. \tag{24}$$

The two other integrals in (8) do not involve the c_r being dependent on p only. Their evaluation is considered in the next section.

5. The contribution of *p*

The calculation of $\partial p_1/\partial y$ on $y = 0$ from the two integrals involving p in (8) is not straightforward because a direct application of the derivative produces singularities in the Green's function. Accordingly, some manipulation is necessary to ensure that convergent integrals are obtained.

Let

$$I_2 = \left[\frac{\partial}{\partial y}\int_0^h p(0, \eta)\left[\frac{\partial}{\partial \xi}G_1(x, y, \xi, \eta)\right]_{\xi=0} \mathrm{d}\eta\right]_{y=0}.$$

There is no difficulty with the series in G_1 when x is positive and so

$$I_2 = \frac{2}{h}\sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}\kappa_n x}\int_0^h p(0, \eta)\nu_n \sin \nu_n \eta \, \mathrm{d}\eta.$$

The main problem now is that a large number of terms has to be calculated when x is small. To avoid this integrate by parts to secure

$$I_2 = \frac{2}{h}p(0, 0)\sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}\kappa_n x} + \frac{2}{h}\sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}\kappa_n x}\int_0^h \frac{\partial p}{\partial \eta}(0, \eta)\cos \nu_n \eta \, \mathrm{d}\eta.$$

Since $\mathrm{i}\kappa_n$ differs by little from ν_n as n increases $\mathrm{e}^{-\nu_n x}$ should be reasonably close to $\mathrm{e}^{-\mathrm{i}\kappa_n x}$ for the later values of n. Also

$$\sum_{n=0}^{\infty}\mathrm{e}^{-\nu_n x}\cos \nu_n \eta = \frac{\sinh(\pi x/2h)\cos(\pi \eta/2h)}{\cosh(\pi x/2h) - \cos(\pi \eta/h)}.$$

Hence

$$I_2 = \frac{2}{h}p(0, 0)\sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{i}\kappa_n x} + \frac{2}{h}\sum_{n=0}^{\infty}(\mathrm{e}^{-\mathrm{i}\kappa_n x} - \mathrm{e}^{-\nu_n x})\int_0^h \frac{\partial p}{\partial \eta}(0, \eta)\cos \nu_n \eta \, \mathrm{d}\eta$$
$$+\frac{2}{h}\sinh(\pi x/2h)\int_0^h \frac{\partial p}{\partial \eta}(0, \eta)\frac{\cos(\pi \eta/2h)}{\cosh(\pi x/h) - \cos(\pi \eta/h)}\mathrm{d}\eta. \tag{25}$$

The only place in (25) where many terms of the series may be involved for small x is in the factor of $p(0, 0)$. It will be seen shortly how this difficulty may be surmounted.

Turn now to

$$I_3 = \left[\frac{\partial}{\partial y} \int_0^\infty p(\xi, 0) \left[\frac{\partial}{\partial \eta} G_1(x, y, \xi, \eta) \right]_{\eta=0} \mathrm{d}\xi \right]_{y=0}.$$

Here the fact that $\xi = x$ lies in the interval of integration is the principal cause of trouble. Consider

$$\frac{1}{\mathrm{i}\kappa_n h} \int_0^x p(\xi, 0)\, \mathrm{e}^{-\mathrm{i}\kappa_n (x-\xi)}\, \mathrm{d}\xi.$$

An integration by parts gives

$$\{p(0,0)\, \mathrm{e}^{-\mathrm{i}\kappa_n x} - p(x,0)\}/\kappa_n^2 h + \frac{1}{h} \int_0^x \frac{\partial p}{\partial \xi}(\xi 0) \left\{ \frac{\mathrm{e}^{-\mathrm{i}\kappa_n (x-\xi)}}{\kappa_n^2} + \frac{\mathrm{e}^{-\nu_n (x-\xi)}}{\nu_n^2} \right\} \mathrm{d}\xi$$

$$-\frac{1}{h\nu_n^2} \int_0^x \frac{\partial}{\partial \xi} p(\xi, 0)\, \mathrm{e}^{-\nu_n (x-\xi)}\, \mathrm{d}\xi.$$

The derivative with respect to η in G_1 multiplies by ν_n. Moreover,

$$\sum_{n=0}^\infty \frac{\nu_n}{\kappa_n^2} \sin \nu_n y = -\frac{h \cos k_1 (y-h)}{2 \cos k_1, h}$$

$$\sum_{n=0}^\infty \frac{\mathrm{e}^{-\nu_n |x|}}{\nu_n} \sin \nu_n y = \frac{h}{2\pi \mathrm{i}} \log \frac{\sinh(\pi |x|/2h) + \mathrm{i} \sin(\pi y/2h)}{\sinh(\pi |x|/2h) - \mathrm{i} \sin(\pi y/2h)},$$

whence

$$\sum_{n=0}^\infty \mathrm{e}^{-\nu_n |x|} \cos \nu_n y = \frac{1}{2} \frac{\sinh(\pi |x|/2h) \cos(\pi y/2h)}{\sinh^2(\pi |x|/2h) + \sin^2(\pi y/2h)},$$

Hence

$$\left[\frac{\partial 2}{\partial y \partial \eta} \sum_{n=0}^\infty \frac{1}{\mathrm{i}\kappa_n h} \int_0^x p(\xi, 0)\, \mathrm{e}^{-\mathrm{i}\kappa_n (x-\xi)} \sin \nu_n y \sin \nu_n \eta\, \mathrm{d}\xi \right]_{\eta=0, y=0}$$

$$= p(0,0) \sum_{n=0}^\infty \frac{\nu_n^2}{\kappa_n^2 h} \mathrm{e}^{-\mathrm{i}\kappa_n x} + \frac{1}{2} p(x,0) k_1 \tan k_1 h + \frac{1}{h} \sum_{n=0}^\infty \int_0^\infty \frac{\partial}{\partial \xi} p(\xi, 0) \left\{ \frac{\nu_n^2}{\kappa_n^2} \mathrm{e}^{-\mathrm{i}\kappa_n (x-\xi)} \right.$$

$$\left. + \mathrm{e}^{-\nu_n (x-\xi)} \right\} \mathrm{d}\xi - \frac{1}{2h} \lim_{y \to 0} \int_0^x \frac{\partial}{\partial \xi} p(\xi, 0) \frac{\sinh(\pi |\xi - x|/2h) \cos(\pi y/2h)}{\sinh^2\{\pi |\xi - x|/2h\} + \sin^2(\pi y/2h)} \mathrm{d}\xi.$$

By repetition of the above process for the remaining integrals in I_3 it is found that

$$I_3 = 2p(0,0) \sum_{n=0}^\infty \frac{\nu_n^2\, \mathrm{e}^{-\mathrm{i}\kappa_n x}}{\kappa_n^2 h} + p(x,0) k_1 \tan k_1 h$$

$$+ \frac{1}{h} \sum_{n=0}^\infty \int_0^x \frac{\partial}{\partial \xi} p(\xi, 0) \left\{ \frac{\nu_n^2}{\kappa_n^2} \mathrm{e}^{-\mathrm{i}\kappa_n (x-\xi)} + \mathrm{e}^{-\nu_n (x-\xi)} \right\} \mathrm{d}\xi$$

$$
\begin{aligned}
&-\frac{1}{h}\sum_{n=0}^{\infty}\int_{x}^{\infty}\frac{\partial}{\partial\xi}p(\xi,0)\left\{\frac{\nu_n^2}{\kappa_n^2}\mathrm{e}^{-\mathrm{i}\kappa_n(\xi-x)}+\mathrm{e}^{-\nu_n(\xi-x)}\right\}\mathrm{d}\xi\\
&+\frac{1}{h}\sum_{n=0}^{\infty}\int_{0}^{\infty}\frac{\partial}{\partial\xi}p(\xi,0)\left\{\frac{\nu_n^2}{\kappa_n^2}\mathrm{e}^{-\mathrm{i}\kappa_n(x+\xi)}+\mathrm{e}^{-\nu_n(x+\xi)}\right\}\mathrm{d}\xi\\
&-\frac{1}{2h}\int_{0}^{\infty}\frac{\partial}{\partial\xi}p(\xi,0)\frac{\mathrm{d}\xi}{\sinh\{\pi(x+\xi)/2h\}}\\
&+\frac{1}{2h}\lim_{y\to 0}\int_{0}^{\infty}\frac{\partial}{\partial\xi}p(\xi,0)\frac{\sinh\{\pi(\xi-x)/2h\}\cos(\pi y/2h)}{\sinh^2\{\pi(\xi-x)/2h\}+\sin^2(\pi y/2h)}\mathrm{d}\xi. \qquad (26)
\end{aligned}
$$

The awkward part of the summation has been isolated now in the last two terms of (26). The remaining series are convergent for all values of the variables with the exception of the factor of $p(0,0)$ as $x\to 0$. Fortunately, when it is added to the corresponding term in (25) the resulting series is convergent, even when x is zero.

For the evaluation of the limit in (26) note that it can be written

$$
\begin{aligned}
&\frac{1}{2h}\int_{0}^{\infty}\left\{\frac{\partial}{\partial\xi}p(\xi,0)-\frac{\partial}{\partial x}p(x,0)\right\}\frac{\mathrm{d}\xi}{\sinh\{\pi(\xi-x)/2h\}}\\
&\quad+\frac{\partial}{\partial x}p(x,0)\lim_{y\to 0}\frac{1}{2h}\int_{0}^{\infty}\frac{\sinh\{\pi(\xi-x)/2h\}\cos(\pi y/2h)}{\sinh^2\{\pi(\xi-x)/2h\}+\sin^2(\pi y/2h)}\mathrm{d}\xi.
\end{aligned}
$$

Inside the limit the integral over the interval (0, 2x) vanishes because the integrand is odd about $\xi=x$. Therefore

$$
\begin{aligned}
&\lim_{y\to 0}\frac{1}{2h}\int_{0}^{\infty}\frac{\partial}{\partial\xi}p(\xi,0)\frac{\sinh\{\pi(\xi-x)/2h\}\cos(\pi y/2h)}{\sinh^2\{\pi(\xi-x)/2h\}+\sin^2(\pi y/2h)}\mathrm{d}\xi\\
&\quad=\frac{1}{2h}\int_{0}^{\infty}\left\{\frac{\partial}{\partial\xi}p(\xi,0)-\frac{\partial}{\partial x}p(x,0)\right\}\frac{\mathrm{d}\xi}{\sinh\{\pi(\xi-x)/2h\}}\\
&\qquad+\frac{1}{2h}\frac{\partial}{\partial x}p(x,0)\int_{2x}^{\infty}\frac{\mathrm{d}\xi}{\sinh\{\pi(\xi-x)/2h\}} \qquad (27)\\
&\quad=\frac{1}{2h}\int_{2x}^{\infty}\frac{\partial}{\partial\xi}p(\xi,0)\frac{\mathrm{d}\xi}{\sinh\{\pi(\xi-x)/2h\}}\\
&\qquad+\frac{1}{2h}\int_{0}^{2x}\left\{\frac{\partial}{\partial\xi}p(\xi,0)-\frac{\partial}{\partial x}p(x,0)\right\}\frac{\mathrm{d}\xi}{\sinh\{\pi(\xi-x)/2h\}}.
\end{aligned}
$$

As a result of these manoeuvres

$$
\left[\frac{\partial p_1}{\partial y}\right]_{y=0}=I_1+I_2+I_3 \qquad (29)
$$

where I_1 is given by (22) or (24) depending on the value of x, I_2 by (25) and I_3 by (26) as modified by (27) or (28). To evaluate the expressions in (29) p needs to be known. Its determination is considered in the next section.

6. An incident plane wave

The function p is specified by (6) where (x_0, y_0) is a point of free space. The formula for the Green's function is expressed most conveniently in terms of polar coordinates. Let $x = r\cos\phi$, $y = r\sin\phi$ where ϕ lies in $(\frac{1}{2}\pi, 2\pi)$ for (x, y) in free space. Likewise let (r_0, ϕ_0) be the polar coordinates of (x_0, y_0).

Define $C_\mu(r, \rho)$ by

$$C_\mu(r, p) = \begin{cases} J_\mu(r)H_\mu^{(2)}(\rho) & (r \le \rho) \\ J_\mu(\rho)H_\mu^2(r) & (r \ge \rho) \end{cases}$$

where J_μ and $H_\mu^{(2)}$ are the standard Bessel and Hankel functions of order μ, respectively. Remark that $C_\mu(r, \rho) = C_\mu(\rho, r)$. Let $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ $(m > 0)$. Then, if $\mu_m = 2m/3$,

$$G_0(x, y, x_0, y_0) = -\frac{\mathrm{i}}{3}\sum_{m=0}^{\infty}\varepsilon_m C_{\mu_m}(kr, kr_0)\cos\frac{2m}{3}\left(\phi - \frac{\pi}{2}\right)\cos\frac{2m}{3}\left(\phi_0 - \frac{\pi}{2}\right) \tag{30}$$

is the Green's function with vanishing normal derivative on the faces of the coating.

To reduce the number of parameters involved and simplify the subsequent analysis it will be assumed now that kr_0 is large. Multiply (30) by $2^{3/2}\mathrm{i}(\pi kr_0)^{1/2}\,\mathrm{e}^{\mathrm{i}(kr_0-\pi/4)}$ and let $kr_0 \to \infty$. The incident field goes over to the plane wave $\mathrm{e}^{\mathrm{i}kr\ \cos(\phi-\phi_0)}$, which has its source in the direction of ϕ_0, and

$$G_0 = \frac{4}{3}\sum_{m=0}^{\infty}\varepsilon_m\ \mathrm{e}^{\frac{1}{2}\mu_m\pi\mathrm{i}}J_{\mu_m}(kr)\cos\frac{2m}{3}\left(\phi - \frac{\pi}{2}\right)\cos\frac{2m}{3}\left(\phi_0 - \frac{\pi}{2}\right).$$

Hence, for the incident wave $\mathrm{e}^{\mathrm{i}kr\ \cos(\phi-\phi_0)}$,

$$p(x, 0) = \frac{4}{3}\sum_{m=0}^{\infty}\varepsilon_m\ \mathrm{e}^{\frac{1}{2}\mu_m\pi\mathrm{i}}(-1)^m J_{\mu_m}(kx)\cos\frac{2m}{3}\left(\phi_0 - \frac{\pi}{2}\right), \tag{31}$$

$$p(0, y) = \frac{4}{3}\sum_{m=0}^{\infty}\varepsilon_m\ \mathrm{e}^{\frac{1}{2}\mu_m\pi\mathrm{i}}J_{\mu_m}(ky)\cos\frac{2m}{3}\left(\phi_0 - \frac{\pi}{2}\right). \tag{32}$$

It is somewhat easier to visualise which face the source of the plane wave is closest to by putting $\phi_0 = \phi_1 + 5\pi/4$. Then $\phi_1 = 0$ corresponds to a source on the line joining the edges of the coating and wedge. As ϕ_1 increases the source moves towards the lower face which lies on $\phi_1 = 3\pi/4$. Negative values of ϕ_1 make the source closer to the upper face which is $\phi_1 = -3\pi/4$. With this change the cosine in (31) and (32) becomes $\cos\{2m(\phi_1 + 3\pi/4)/3\}$.

Each term in the series (31) or (32) can be dealt with separately. Suppose we take $p(x, 0) = (-1)^m J_{\mu_m}(kx)$ and $p(0, y) = J_{\mu_m}(ky)$. Then $f(x)$ is known from (11) and then (21) provides equations to determine the corresponding c_r. Consequently I_1 can be calculated from (22) and (24); denote its value by $I_1(\mu_m)$. Define $I_2(\mu_m)$ and $I_3(\mu_m)$ in a similar fashion. It is evident from (22) that $I_1(\mu_m)$ becomes exponentially small as $x \to \infty$. The formula (25) reveals that $I_2(\mu_m)$ is exponentially small too. With regard to $I_3(\mu_m)$ it is clear, when account is taken of (28), that the integrals provide a small contribution. Hence the term in p(x,0) dominates with the result

$$\frac{\partial p_1}{\partial y} \sim p(x,0)k_1 \tan k_1 h$$

as $x \to \infty$ on $y = 0$.

Let Z be the normalised surface impedance presented to the exterior field on the lower face of the coating. Then

$$Z = \frac{\mathrm{i}}{k}\frac{\partial p/\partial y}{p} = \frac{\mathrm{i}k}{k_y^2}\frac{\partial p_1/\partial y}{p}$$

by virtue of (3). Thus

$$Z \sim (\mathrm{i}k/k_1)\tan k_1 h \tag{33}$$

as $x \to \infty$.

Next the case when $x \to 0$ is examined. The Bessel functions are finite at the origin. So are their derivatives apart from $J_{2/3}$. Hence the only term which can supply singular behaviour at the origin arises from $J_{2/3}$. It is not difficult to show from the preceding formulae that, if only singular terms are retained,

$$Z \approx \frac{\mathrm{i}k^{5/3}2^{1/3}3^{1/2}\,\mathrm{e}^{\pi\mathrm{i}/3}}{(-\frac{1}{3})!\,k_1^2 x^{1/3}}\cos\frac{2}{3}\left(\phi_1 + \frac{3\pi}{4}\right) \tag{34}$$

as $x \to 0$. Thus Z becomes unbounded at the edge of the coating for all positions of the source except one. The exception is $\phi_1 = 0$ when the singularity disappears, *i.e.* Z is finite at the edge when the structure is excited symmetrically. This is in accord with the predictions of static considerations in the neighbourhood of the edge.

In general

$$Z = \frac{4\mathrm{i}k}{3k_1^2 p}\sum_{m=0}^{\infty}\varepsilon_m\,\mathrm{e}^{\frac{1}{2}\mu_m\pi\mathrm{i}}\{I_1(\mu_m) + I_2(\mu_m) + I_3(\mu_m)\}\cos\frac{2m}{3}\left(\phi_i + \frac{3\pi}{4}\right) \tag{35}$$

where p is given by (31) with the change from ϕ_0 to ϕ_1. The advantage of this form from a computational point of view is that $I_1(\mu_m)$, $I_2(\mu_m)$ and $I_3(\mu_m)$ need to be calculated only once and then stored. They do not vary with the position of the source.

Equations (33) and (34) are estimates of the limiting behaviour of (35). It is not by any means transparent from the foregoing discussion how large x must be for Z to assume a constant value for which (33) offers a first approximation. An example in the next section suggests that the transition takes place within a free-space wavelength from the edge of the coating.

7. An example

To illustrate the preceding theory Z has been calculated from (35) for a coating in which $k_1/k = (10 - 10\mathrm{i})^{1/2} = 3{\cdot}47434 - 1{\cdot}43912\mathrm{i}$. This value is the same as that employed for the impedance when the perfectly conducting wedge is absent (see Jones [1, 2]). The thickness of the coating is specified by $h = 0{\cdot}08\lambda$ where λ is the free-space wavelength. With these parameters the asymptotic estimate of (33) gives

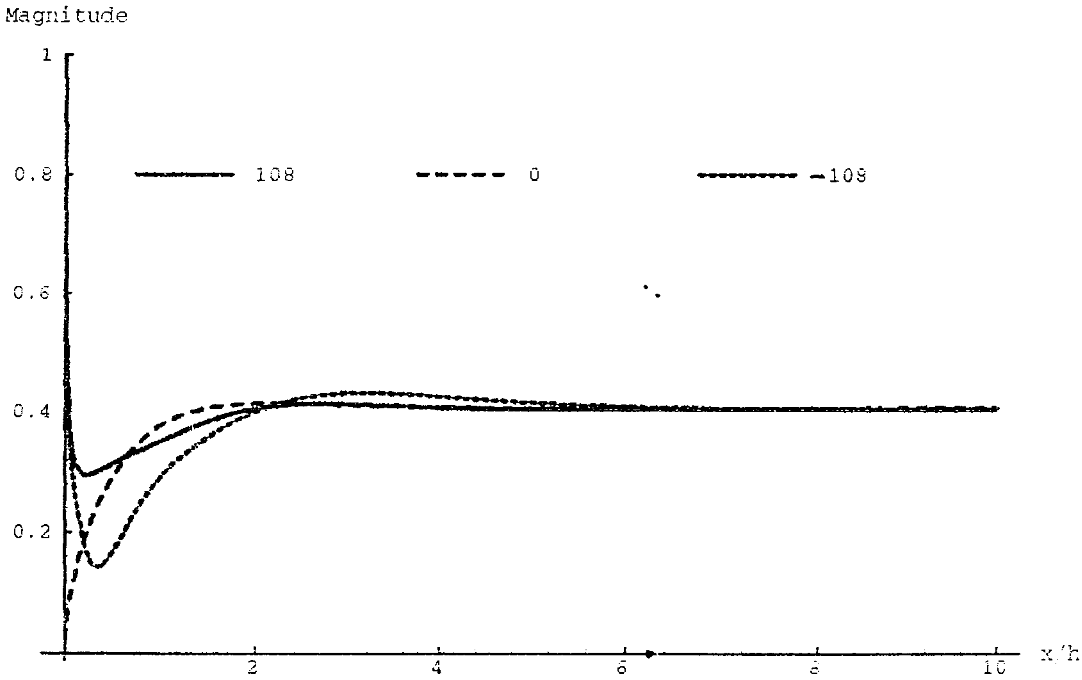

Figure 1. Magnitude of impedance when $k_1/k = (10 - 10\mathrm{i})^{1/2}$, $h = 0{\cdot}08\lambda$ for $\phi_1 = -108°, 0°, 108°$.

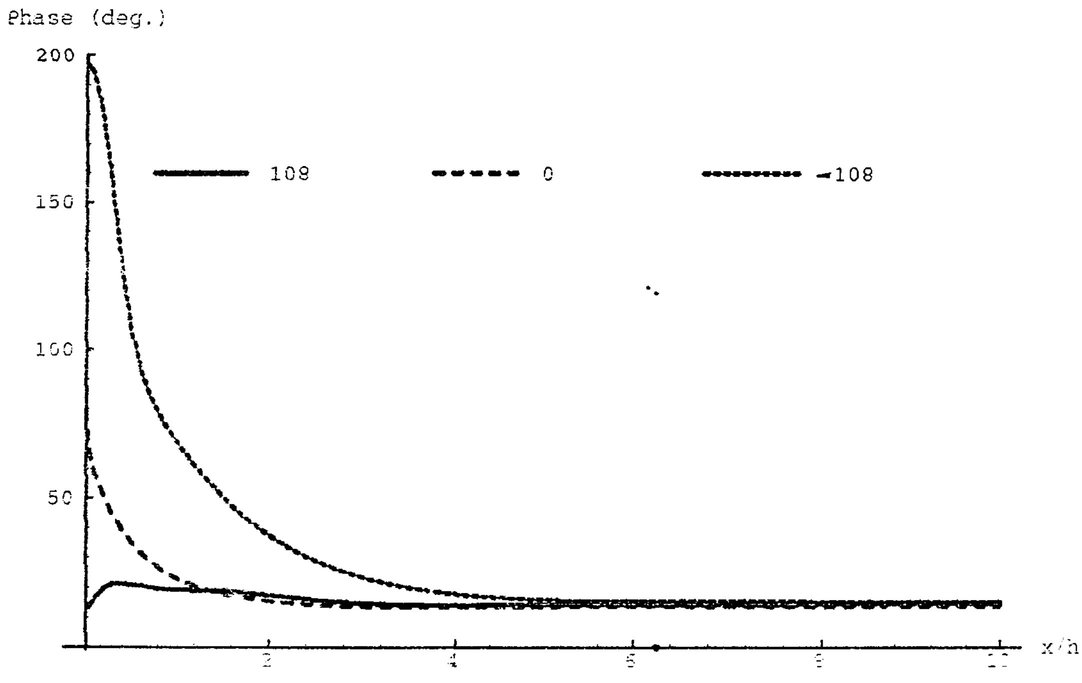

Figure 2. Phase of impedance when $k_1/k = (10 - 10\mathrm{i})^{1/2}$, $h = 0{\cdot}08\lambda$ for $\phi_1 = -108°, 0°, 108°$.

$$|Z| \sim 0{\cdot}4154, \quad \mathrm{ph}(Z) \sim 12{\cdot}77^\circ. \tag{36}$$

Graphs of the magnitude and phase of Z have been drawn in Figures 1 and 2 for various positions of the source. The solid curve corresponds to $\phi_1 = 108^\circ$. For this angle of incidence the lower face is illuminated by the incident plane wave, while the upper face is in shadow. The dashed curve is for $\phi_1 = 0^\circ$; in this case the source occupies a position on the line joining the two edges and both faces are illuminated symmetrically. The remaining (dotted) curve gives Z for $\phi_1 = -108^\circ$; now the upper face is illuminated but the lower face is in shadow. Although Z is given only on the lower face, its value on the upper face can be inferred by considerations of symmetry. Hence, for these angles, the surface impedance is known over the entire boundary of the coating.

On the graphs the horizontal coordinate indicates the distance from the edge of the coating in terms of the thickness. In order to facilitate comparison with the free-space wavelength, a dot has been placed at a distance of $\lambda/2$ from the edge of the coating.

It can be seen that in all the graphs Z has settled down to a constant value well before x reaches $10h$. Since $10h < \lambda$, the transition to constant impedance occurs within a free-space wavelength of the edge of the coating. It does not matter whether the face is in shadow or not. The magnitude of the constant impedance is in excellent agreement with the estimate of (36). The phase is not quite so close, there being differences of a degree or two. Nevertheless, the agreement is good enough to regard (33) as a tolerable approximation to Z as soon as one is a wavelength from the edge.

On the whole the magnitude settles down faster than the phase. As soon as one is two thicknesses from the edge the magnitude is fairly near its final value, whereas the phase may require four or more thicknesses before approaching its terminal constancy. The curve of the magnitude is similar for all excitations except the symmetrical. From a largish value near the edge it falls to a minimum and then rises to its asymptotic value. The phase exhibits more variation between the illuminated and shadow sides. On the illuminated side it rises to a maximum before dropping to its final value whereas in the shadow it falls steadily as the distance from the edge increases.

8. Conclusion

A method has been devised for finding the surface impedance of a coated right-angled wedge irradiated by a TE plane wave. The theory holds provided that the coating is of high contrast and dissipative. In addition, the wedge must be perfectly conducting. Over much of each face of the coating the surface impedance is constant and the same as if the coating were backed by a perfectly conducting infinite plane. This statement applies and the constant is the same whether the face is illuminated or in the shadow. The impedance does vary within the vicinity of the edge of the coating. Probably the region of variability does not extend beyond a free-space wavelength from the edge so long as the contrast and absorption of the coating are high enough.

References

1. D. S. Jones, Impedance of an absorbing wedge. *Quart J. Mech. appl. Math.* 54 (2001) 257–271.
2. D. S. Jones, Impedance of a lossy wedge. *IMA J. App. Math.* 66 (2001) 411–422.